Math 10B with Professor Stankova
Worksheet, Discussion #10; Tuesday, 2/26/2019
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1 Bayes Theorem

1.1 Concepts

1. We use **Bayes theorem** when we want to find the probability of A given B but we are told the opposite probability, the probability of B given A. There are several forms of Bayes Theorem as follows:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{1}{1 + \frac{P(B|\bar{A})P(\bar{A})}{P(B|A)P(A)}}.$$

In order to discern which form to use, look at the information you are given. If you are told $P(B|A)$ as well as $P(B|\bar{A})$, use the latter two methods but if you are only told $P(B)$, then use the first form.

We say that two events A, B are **independent** if $P(A \cap B) = P(A)P(B)$.

1.2 Examples

2. There are 10 red and 10 blue balls in a bag. Someone randomly picks out a ball and then places it back and puts 10 more balls of that color into the bag. Then you draw a ball. What is the probability that the 10 balls added were red, given that you drew out a red ball?

Solution: We use Bayes Theorem to get

$$P(AddRed|DrawRed) = \frac{1}{1 + \frac{P(DrawRed|AddBlue)P(AddBlue)}{P(DrawRed|AddRed)P(AddRed)}}$$

$$= \frac{1}{1 + \frac{10/30 \cdot 1/2}{20/30 \cdot 1/2}} = \frac{1}{1 + 1/2} = \frac{2}{3}.$$

3. Out of those brought to court, there are 60% which are actually guilty. Of those that are guilty, 95% of them are convicted. But there are 1% of innocent people who get falsely convicted. What is the probability that you are actually innocent given that you are convicted?

Solution: We use Bayes rule which tells us that $P(Innocent|Convicted)$

$$= \frac{P(Convicted|Innocent)P(Innocent)}{P(Convicted|Innocent)P(Innocent) + P(Convicted|Guilty)P(Guilty)}$$

$$= \frac{0.01 \cdot 0.4}{0.01 \cdot 0.4 + 0.95 \cdot 0.6} \approx 0.7\%$$

1.3 Problems

4. True **FALSE** We can always use the formula $P(A|B) = \frac{1}{1+\frac{P(B|\bar{A})P(\bar{A})}{P(B|A)P(A)}}$.

Solution: This is undefined if $P(B) = 0$.

5. I have two boxes of apples and oranges. In box 1, there are 5 oranges and 6 apples, in box 2 there are 6 oranges and 5 apples. I randomly pick a box and then in this box randomly pick a fruit. What is the probability that I picked box 1 given that I picked an orange?

Solution: We have that

$$P(Box1|Orange) = \frac{P(Box1 \cap Orange)}{P(Orange)}$$

$$= \frac{P(Orange|Box1)P(Box1)}{P(Orange|Box1)P(Box1) + P(Orange|Box2)P(Box2)} = \frac{5/11 \cdot 1/2}{5/11 \cdot 1/2 + 6/11 \cdot 1/2} = \frac{5}{11}.$$

6. An exam has a 99% chance of testing positive if you have the disease and 1% chance of testing positive if you do not have the disease. Give that 0.5% of people have this disease, what is the probability that you have the disease given that you tested positive?

Solution:

$$P(Sick|Positive) = \frac{P(Positive|Sick)P(Sick)}{P(Positive|Sick)P(Sick) + P(Positive|Notsick)P(Notsick)}$$

$$= \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.33.$$

7. About 2/3 of drivers use their cell phone while driving. Suppose that you are 5 times more likely to get into an accident if you text and drive, and if you don't use your cell phone, you have a 1% chance of getting into an accident. What is the probability that someone was texting given that they got into an accident?

Solution: We have that

$$P(Text|Accident) = \frac{P(Accident|Text)P(Text)}{P(Accident|Text)P(Text) + P(Accident|NoText)P(NoText)}$$

$$= \frac{0.05 \cdot 2/3}{0.05 \cdot 2/3 + 0.01 \cdot 1/3} = \frac{10}{11}.$$

2 Review

8. How many ways can 10 boys be paired up with 10 girls so that each boy is paired up with one girl.

Solution: The first boy has 10 choices, the second now has 9, and so on so $10 \cdot 9 \cdots 1 = 10!$.

9. How many ways can you arrange 10 marbles in a row if 4 are red, 3 are blue, and 3 are green (marbles of the same color are identical)?

Solution:

$$\binom{10}{4}\binom{6}{3} = \frac{10!}{4!3!3!}.$$

10. Prove that $\binom{n}{r} = \binom{n}{n-r}$ in two different ways.

Solution: We can first prove it algebraically by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}.$$

Then, to prove it combinatorially, the left side is the number of ways to choose a team of r people out of n and this is the same as choosing who is NOT on the team ($n - r$ people).

11. Zvezda and Ramanujan play a game. They roll 4 6 sided die. If at least one 6 is rolled, then Zvezda wins. What is the probability that she wins?

Solution: The total number of ways the game can go is 6^4. She wins if at least one 6 is rolled, which is 6^4 minus the number of ways no 6's are rolled. So, she wins with probability

$$1 - \frac{5^4}{6^4}.$$

12. How many solutions are there to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$

if $x_1 \geq 4, x_2 \geq 8, x_3 \geq 2, x_4 \geq 3, x_5 \geq 0$?

Solution: First we subtract to get rid of the restrictions to get $100-4-8-2-3-0 = 83$ balls and 5 boxes. So the answer is

$$\binom{83+5-1}{83} = \binom{87}{83}.$$

13. How many subsets of $\{1, 2, \ldots, n\}$ contain at least one of 1 and 2.

Solution: We use PIE. There are 2^{n-1} subsets that contain 1, 2^{n-1} that contain 2, and 2^{n-2} that contain both. So the answer is

$$2^{n-1} + 2^{n-1} - 2^{n-2} = 2^n - 2^{n-2}.$$

14. Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1$ and $a_{n+1} = 4a_n + 1$. Prove that $a_n = \frac{4^{n+1}-1}{3}$ for all $n \geq 0$.

Solution: By induction on n.
Base Case ($n = 0$): Note $a_0 = 1 = \frac{4^1-1}{3}$.
Inductive Hypothesis: Assume $a_n = \frac{4^{n+1}-1}{3}$ for some $n \geq 0$. Inductive Step: Then $a_{n+1} = 4a_n + 1 = 4\frac{4^{n+1}-1}{3} + 1 = \frac{4^{n+2}-4+3}{3} = \frac{4^{n+2}-1}{3}$.
Thus by induction $a_n = \frac{4^{n+1}-1}{3}$ for all $n \geq 0$.

15. Prove that if you select $n+1$ distinct numbers from 1 to $2n$, then at least two of the numbers sum to $2n+1$.

Solution: We pair them up as $(1,2n),(2,2n-1),\dots,(n,n+1)$. There are n pairs so by pigeonhole, we must select two in the same pair which means we have two that sum to $2n+1$.

16. (Challenge) How many ways are there to sit 3 males and 7 females at a circular table so that no two males sit next to each other? (HINT: First try to do this problem when we only care about the order of M and F)

Solution: First let's assume that the males are all identical and females are identical and we just consider order of males and females. We pair up the males with a female so we need to seat 3 pairs of MF and 4 remaining 4 Fs. There are $\frac{7!}{3!4!}\cdot\frac{1}{7}$ ways to do this (we divide by 7 because rotating the table gives the same seating arrangement). Finally, once we determine the ordering of males and females, we multiply by 3! for the ways the three males to sit and 7! for the females to get

$$\frac{7!}{3!4!}\frac{3!7!}{7}=\frac{7!6!}{4!}.$$